Contact Integrable Extensions and Zero-Curvature Representations for the Second Heavenly Equation

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Abstract. The method of contact integrable extensions is used to find new zero-curvature representation for Plebański's second heavenly equation.

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1. Introduction

The second Plebañski's heavenly equation, [27],

$$u_{xz} = u_{ty} + u_{yy}\,u_{zz} - u_{yz}^2, \tag{1}$$

describes self-dual gravitational fields. This equation can be obtained as the compatibility condition for the following system of PDEs, [10, 1], cf. [27, Eq. (3.13)],

$$\left\{\begin{array}{l} v_t = (u_{yz} + \lambda)\,v_z - u_{zz}\,v_y, \\ v_x = u_{yy}\,v_z - (u_{yz} - \lambda)\,v_y \end{array}\right. \tag{2}$$

with an arbitrary constant λ. This condition is equivalent to the commutativity of four infinite-dimensional vector fields

$$\begin{aligned}
\widetilde{D}_t &= \bar{D}_t + \sum_{i,j\ge 0} \widetilde{D}_y^i \widetilde{D}_z^j\left((u_{yz}+\lambda)\,v_{0,1} - u_{zz}\,v_{1,0}\right)\frac{\partial}{\partial v_{i,j}},\\
\widetilde{D}_x &= \bar{D}_x + \sum_{i,j\ge 0} \widetilde{D}_y^i \widetilde{D}_z^j\left(u_{yy}\,v_{0,1} - (u_{yz}-\lambda)\,v_{1,0}\right)\frac{\partial}{\partial v_{i,j}},\\
\widetilde{D}_y &= \bar{D}_y + \sum_{i,j\ge 0} v_{i+1,j}\,\frac{\partial}{\partial v_{i,j}},\\
\widetilde{D}_z &= \bar{D}_z + \sum_{i,j\ge 0} v_{i,j+1}\,\frac{\partial}{\partial v_{i,j}},
\end{aligned}$$

where $\bar{D}_t$, $\bar{D}_x$, $\bar{D}_y$ and $\bar{D}_z$ are restrictions of the total derivatives D_t, D_x, D_y and D_z to the infinite prolongation of Eq. (1). This construction is called a *differential covering*, [15] – [18], or zero-curvature representation. Dually Eqs. (2) can be defined by means of differential 1-form

$$\omega = dv + (v_{zz}\,v_y - (u_{yz}+\lambda)\,v_z)\,dt + ((u_{yz}-\lambda)\,v_y - u_{yy}\,v_z)\,dx - v_y\,dy - v_z\,dz \tag{3}$$

called the *Wahlquist–Estabrook form* of the covering, [9]. In [25] we show that this form can be inferred from a linear combination of Maurer–Cartan forms of the contact symmetry pseudo-group of Eq. (1). In this paper we apply to (1) the technique of contact integrable extensions (CIEs) proposed in [24]. We find CIEs of the structure equations of the contact symmetry pseudo-group of Eq. (1). The analysis of these CIEs splits into two cases. In the first case integration of the CIE gives Eqs. (2), while in the second case we obtain new covering of the second heavenly equation.

2. Symmetry pseudo-group of the second heavenly equation

Let $\pi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a vector bundle with the local base coordinates $(x^1, ..., x^n)$ and the local fibre coordinate u; then denote by $J^2(\pi)$ the bundle of the second-order jets of sections of π, with the local coordinates (x^i, u, u_i, u_{ij}), $i, j \in \{1, ..., n\}$, $i \le j$. For every local section $(x^i, f(x))$ of π, denote by $j_2(f)$ the corresponding 2-jet $(x^i, f(x), \partial f(x)/\partial x^i, \partial^2 f(x)/\partial x^i \partial x^j)$. A differential 1-form ϑ on $J^2(\pi)$ is called a *contact form* if it is annihilated by all 2-jets of local sections: $j_2(f)^*\vartheta = 0$. In the local

coordinates every contact 1-form is a linear combination of the forms $\vartheta_0 = du - u_i\,dx^i$, $\vartheta_i = du_i - u_{ij}\,dx^j$, $i, j \in \{1, ..., n\}$, $u_{ji} = u_{ij}$ (here and later we assume the summation convention, so $u_i\,dx^i = \sum_{i=1}^{n} u_i\,dx^i$, etc.) A local diffeomorphism $\Delta\colon J^2(\pi) \to J^2(\pi)$, $\Delta\colon (x^i, u, u_i, u_{ij}) \mapsto (\check{x}^i, \check{u}, \check{u}_i, \check{u}_{ij})$, is called a *contact transformation* if for every contact 1-form $\check{\vartheta}$ the form $\Delta^*\check{\vartheta}$ is also contact. We denote by $\mathrm{Cont}(J^2(\pi))$ the pseudo-group of contact transformations on $J^2(\pi)$.

Let $\mathcal{H} \subset \mathbb{R}^{(2n+1)(n+3)(n+1)/3}$ be an open set with local coordinates a, b^i_k, c^i, f^{ik}, g_i, s_{ij}, w^k_{ij}, u_{ijk}, $i, j, k \in \{1, ..., n\}$, such that $a \neq 0$, $\det(b^i_k) \neq 0$, $f^{ik} = f^{ki}$, $u_{ijk} = u_{jik} = u_{ikj}$. Let (B^i_k) be the inverse matrix for the matrix (b^k_l), so $B^i_k\,b^k_l = \delta^i_l$. We consider the *lifted coframe*

$$
\begin{aligned}
&\Theta_0 = a\,\vartheta_0, \quad \Theta_i = g_i\,\Theta_0 + a\,B^k_i\,\vartheta_k, \quad \Xi^i = c^i\,\Theta_0 + f^{ik}\,\Theta_k + b^i_k\,dx^k,\\
&\Theta_{ij} = a\,B^k_i\,B^l_j\,(du_{kl} - u_{klm}\,dx^m) + s_{ij}\,\Theta_0 + w^k_{ij}\,\Theta_k,
\end{aligned}
\tag{4}
$$

$i \le j$, defined on $J^2(\pi) \times \mathcal{H}$. As it is shown in [21], the forms (4) are Maurer–Cartan forms for $\mathrm{Cont}(J^2(\pi))$, that is, a local diffeomorphism $\widehat{\Delta}\colon J^2(\pi)\times\mathcal{H} \to J^2(\pi)\times\mathcal{H}$ satisfies the conditions $\widehat{\Delta}^*\,\check{\Theta}_0 = \Theta_0$, $\widehat{\Delta}^*\,\check{\Theta}_i = \Theta_i$, $\widehat{\Delta}^*\,\check{\Xi}^i = \Xi^i$, and $\widehat{\Delta}^*\,\check{\Theta}_{ij} = \Theta_{ij}$ whenever it is projectable on $J^2(\pi)$, and its projection $\Delta\colon J^2(\pi) \to J^2(\pi)$ is a contact transformation.

The structure equations for $\mathrm{Cont}(J^2(\pi))$ read

$$
\begin{aligned}
d\Theta_0 &= \Phi^0_0 \wedge \Theta_0 + \Xi^i \wedge \Theta_i,\\
d\Theta_i &= \Phi^0_i \wedge \Theta_0 + \Phi^k_i \wedge \Theta_k + \Xi^k \wedge \Theta_{ik},\\
d\Xi^i &= \Phi^0_0 \wedge \Xi^i - \Phi^i_k \wedge \Xi^k + \Psi^{i0} \wedge \Theta_0 + \Psi^{ik} \wedge \Theta_k,\\
d\Theta_{ij} &= \Phi^k_i \wedge \Theta_{kj} - \Phi^0_0 \wedge \Theta_{ij} + \Upsilon^0_{ij} \wedge \Theta_0 + \Upsilon^k_{ij} \wedge \Theta_k + \Xi^k \wedge \Theta_{ijk},
\end{aligned}
$$

where the additional forms Φ^0_0, Φ^0_i, Φ^k_i, Ψ^{i0}, Ψ^{ij}, Υ^0_{ij}, Υ^k_{ij}, and Θ_{ijk} depend on differentials of the coordinates of $\mathcal{H}$.

Suppose $\mathcal{E}$ is a second-order differential equation in one dependent and n independent variables. We consider $\mathcal{E}$ as a submanifold in $J^2(\pi)$. Let $\mathrm{Cont}(\mathcal{E})$ be the group of contact symmetries for $\mathcal{E}$. It consists of all the contact transformations on $J^2(\pi)$ mapping $\mathcal{E}$ to itself. Let $\iota_0\colon \mathcal{E} \to J^2(\pi)$ be an embedding and $\iota = \iota_0 \times \mathrm{id}\colon \mathcal{E}\times\mathcal{H} \to J^2(\pi) \times \mathcal{H}$. Maurer–Cartan forms of the pseudo-group $\mathrm{Cont}(\mathcal{E})$ can be obtained from the forms $\theta_0 = \iota^*\Theta_0$, $\theta_i = \iota^*\Theta_i$, $\xi^i = \iota^*\Xi^i$ and $\theta_{ij} = \iota^*\Theta_{ij}$ by means of Élie Cartan's method of equivalence, [3]–[6], [8, 13, 26], see details and examples in [7], [20]—[25].

Using this method, we find the Maurer–Cartan forms and their structure equations for the symmetry pseudo-group of Eq. (1). The structure equations have the following form:

$$
d\theta_0 = \eta_5 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2 + \xi^3 \wedge \theta_3 + \xi^4 \wedge \theta_4,
$$

$$
\begin{aligned}
d\theta_1 = {}&(\eta_5 - \eta_1) \wedge \theta_1 - \eta_3 \wedge \theta_2 - \theta_{44} \wedge \theta_3 + \eta_6 \wedge \theta_4 + \xi^1 \wedge \theta_{11} + \xi^2 \wedge \theta_{12} + \xi^3 \wedge \theta_{13}\\
&+\xi^4 \wedge \theta_{14},
\end{aligned}
$$

$$
d\theta_2 = -\eta_2 \wedge \theta_1 + (\eta_5 - \eta_4) \wedge \theta_2 + (\eta_6 - 2\,\theta_{34}) \wedge \theta_3 + \theta_{33} \wedge \theta_4 + \xi^1 \wedge \theta_{12} + \xi^2 \wedge \theta_{22}
$$

$$+\xi^3 \wedge \theta_{23} + \xi^4 \wedge \theta_{13},$$

$$d\theta_3 = \tfrac{1}{3}\,(\eta_1 - 2\,\eta_4 - 2\,\eta_5) \wedge \theta_3 - \eta_2 \wedge \theta_4 + \xi^1 \wedge \theta_{13} + \xi^2 \wedge \theta_{23} + \xi^3 \wedge \theta_{33} + \xi^4 \wedge \theta_{34},$$

$$d\theta_4 = -\eta_3 \wedge \theta_3 + \tfrac{1}{3}\,(\eta_4 - 2\,\eta_1 + 2\,\eta_5) \wedge \theta_4 + \xi^1 \wedge \theta_{14} + \xi^2 \wedge \theta_{13} + \xi^3 \wedge \theta_{34} + \xi^4 \wedge \theta_{44},$$

$$d\xi^1 = \eta_1 \wedge \xi^1 + \eta_2 \wedge \xi^2,$$

$$d\xi^2 = \eta_3 \wedge \xi^1 + \eta_4 \wedge \xi^2,$$

$$d\xi^3 = \theta_{44} \wedge \xi^1 - (\eta_6 - 2\,\theta_{34}) \wedge \xi^2 + \tfrac{1}{3}\,(\eta_5 - \eta_1 + 2\,\eta_4) \wedge \xi^3 + \eta_3 \wedge \xi^4,$$

$$d\xi^4 = -\eta_6 \wedge \xi^1 - \theta_{33} \wedge \xi^2 + \eta_2 \wedge \xi^3 + \tfrac{1}{3}\,(\eta_5 + 2\,\eta_1 - \eta_4) \wedge \xi^4,$$

$$d\theta_{11} = -\eta_{12} \wedge \theta_1 + \eta_{14} \wedge \theta_2 + \eta_{13} \wedge \theta_3 - \eta_{11} \wedge \theta_4 + \eta_5 \wedge \theta_{11} - 2\,\eta_1 \wedge \theta_{11} - 2\,\eta_3 \wedge \theta_{12}$$
$$-2\,\theta_{44} \wedge \theta_{13} + 2\,\eta_6 \wedge \theta_{14} - \xi^1 \wedge \eta_{22} - \xi^2 \wedge \eta_{21} - \xi^3 \wedge \eta_{17} - \xi^4 \wedge \eta_{18},$$

$$d\theta_{12} = -\eta_{10} \wedge \theta_1 + \eta_{12} \wedge \theta_2 + \eta_{11} \wedge \theta_3 - \eta_7 \wedge \theta_4 - \eta_2 \wedge \theta_{11} + (\eta_5 - \eta_1 - \eta_4) \wedge \theta_{12}$$
$$+2\,(\eta_6 - \theta_{34}) \wedge \theta_{13} + \theta_{33} \wedge \theta_{14} - \eta_3 \wedge \theta_{22} - \theta_{44} \wedge \theta_{23} - \xi^1 \wedge \eta_{21} - \xi^2 \wedge \eta_{19}$$
$$-\xi^3 \wedge \eta_{15} - \xi^4 \wedge \eta_{17},$$

$$d\theta_{13} = \eta_{12} \wedge \theta_3 - \eta_{10} \wedge \theta_4 + \tfrac{2}{3}\,(\eta_5 - \eta_1 - \eta_4) \wedge \theta_{13} - \eta_2 \wedge \theta_{14} - \eta_3 \wedge \theta_{23} - \theta_{44} \wedge \theta_{33}$$
$$+\eta_6 \wedge \theta_{34} - \xi^1 \wedge \eta_{17} - \xi^2 \wedge \eta_{15} - \xi^3 \wedge \eta_7 - \xi^4 \wedge \eta_{11},$$

$$d\theta_{14} = \eta_{14} \wedge \theta_3 - \eta_{12} \wedge \theta_4 - 2\,\eta_3 \wedge \theta_{13} + \tfrac{1}{3}\,(\eta_4 - 5\,\eta_1 + 2\,\eta_5) \wedge \theta_{14} + (\eta_6 + \theta_{34}) \wedge \theta_{44}$$
$$-\xi^1 \wedge \eta_{18} - \xi^2 \wedge \eta_{17} - \xi^3 \wedge \eta_{11} - \xi^4 \wedge \eta_{13},$$

$$d\theta_{22} = -\eta_9 \wedge \theta_1 + \eta_{10} \wedge \theta_2 + \eta_7 \wedge \theta_3 - \eta_8 \wedge \theta_4 - 2\,\eta_2 \wedge \theta_{12} + 2\,\theta_{33} \wedge \theta_{13} - \xi^1 \wedge \eta_{19}$$
$$+(\eta_5 - 2\,\eta_4) \wedge \theta_{22} + 2\,(\eta_6 - 2\,\theta_{34}) \wedge \theta_{23} - \xi^2 \wedge \eta_{20} - \xi^3 \wedge \eta_{16} - \xi^4 \wedge \eta_{15},$$

$$d\theta_{23} = \eta_{10} \wedge \theta_3 - \eta_9 \wedge \theta_4 - 2\,\eta_2 \wedge \theta_{13} + \tfrac{1}{3}\,(\eta_1 - 5\,\eta_4 + 2\,\eta_5) \wedge \theta_{23} + (\eta_6 - 3\,\theta_{34}) \wedge \theta_{33}$$
$$-\xi^1 \wedge \eta_{15} - \xi^2 \wedge \eta_{16} - \xi^3 \wedge \eta_8 - \xi^4 \wedge \eta_7,$$

$$d\theta_{33} = \tfrac{1}{3}\,(\eta_5 + 2\,\eta_1 - 4\,\eta_4) \wedge \theta_{33} - 2\,\eta_2 \wedge \theta_{34} - \xi^1 \wedge \eta_7 - \xi^2 \wedge \eta_8 - \xi^3 \wedge \eta_9 - \xi^4 \wedge \eta_{10},$$

$$d\theta_{34} = -\eta_3 \wedge \theta_{33} + \tfrac{1}{3}\,(\eta_5 - \eta_1 - \eta_4) \wedge \theta_{34} - \eta_2 \wedge \theta_{44} - \xi^1 \wedge \eta_{11} - \xi^2 \wedge \eta_7 - \xi^3 \wedge \eta_{10}$$
$$-\xi^4 \wedge \eta_{12},$$

$$d\theta_{44} = -2\,\eta_3 \wedge \theta_{34} + \tfrac{1}{3}\,(\eta_5 - 4\,\eta_1 + 2\,\eta_4) \wedge \theta_{44} - \xi^1 \wedge \eta_{13} - \xi^2 \wedge \eta_{11} - \xi^3 \wedge \eta_{12} - \xi^4 \wedge \eta_{14},$$

$$d\eta_1 = \eta_2 \wedge \eta_3 - \eta_{12} \wedge \xi^1 - \eta_{10} \wedge \xi^2,$$

$$d\eta_2 = (\eta_1 - \eta_4) \wedge \eta_2 - \eta_{10} \wedge \xi^1 - \eta_9 \wedge \xi^2,$$

$$d\eta_3 = (\eta_4 - \eta_1) \wedge \eta_3 + \eta_{14} \wedge \xi^1 + \eta_{12} \wedge \xi^2,$$

$$d\eta_4 = -\eta_2 \wedge \eta_3 + \eta_{12} \wedge \xi^1 + \eta_{10} \wedge \xi^2,$$

$d\eta_5 = 0,$

$d\eta_6 = \frac{1}{3}\,(\eta_5 - \eta_1 - \eta_4) \wedge \eta_6 - \eta_3 \wedge \theta_{33} - \eta_2 \wedge \theta_{44} + \eta_{11} \wedge \xi^1 + \eta_7 \wedge \xi^2 + \eta_{10} \wedge \xi^3$
$\quad + \eta_{12} \wedge \xi^4,$

$d\eta_7 = \frac{1}{3}\,(\eta_5 - \eta_1 - \eta_4) \wedge \eta_7 - 2\,\eta_2 \wedge \eta_{11} - \eta_3 \wedge \eta_8 + \eta_6 \wedge \eta_{10} - 2\,\eta_{12} \wedge \theta_{33} + 2\,\eta_{10} \wedge \theta_{34}$
$\quad + \eta_9 \wedge \theta_{44} + \eta_{23} \wedge \xi^1 + \eta_{24} \wedge \xi^2 + \eta_{25} \wedge \xi^3 + \eta_{26} \wedge \xi^4,$

$d\eta_8 = \frac{1}{3}\,(\eta_5 + 2\,\eta_1 - 7\,\eta_4) \wedge \eta_8 - 3\,\eta_2 \wedge \eta_7 + \eta_6 \wedge \eta_9 - 3\,\eta_{10} \wedge \theta_{33} + 4\,\eta_9 \wedge \theta_{34} + \eta_{24} \wedge \xi^1$
$\quad + \eta_{27} \wedge \xi^2 + \eta_{28} \wedge \xi^3 + \eta_{25} \wedge \xi^4,$

$d\eta_9 = (\eta_1 - 2\,\eta_4) \wedge \eta_9 - 3\,\eta_2 \wedge \eta_{10} + \eta_{25} \wedge \xi^1 + \eta_{28} \wedge \xi^2,$

$d\eta_{10} = -2\,\eta_2 \wedge \eta_{12} - \eta_3 \wedge \eta_9 - \eta_4 \wedge \eta_{10} + \eta_{26} \wedge \xi^1 + \eta_{25} \wedge \xi^2,$

$d\eta_{11} = \frac{1}{3}\,(\eta_5 - \eta_1 - \eta_4) \wedge \eta_{11} - \eta_2 \wedge \eta_{13} - 2\,\eta_3 \wedge \eta_7 + \eta_6 \wedge \eta_{12} + \eta_{29} \wedge \xi^1 + \eta_{23} \wedge \xi^2$
$\quad + \eta_{26} \wedge \xi^3 + \eta_{30} \wedge \xi^4 + \eta_{14} \wedge \theta_{33} + 2\,\eta_{10} \wedge \theta_{44},$

$d\eta_{12} = -\eta_1 \wedge \eta_{12} - \eta_2 \wedge \eta_{14} - 2\,\eta_3 \wedge \eta_{10} + \eta_{30} \wedge \xi^1 + \eta_{26} \wedge \xi^2,$

$d\eta_{13} = \frac{1}{3}\,(\eta_5 - 7\,\eta_1 + 2\,\eta_4) \wedge \eta_{13} - 3\,\eta_3 \wedge \eta_{11} + (\eta_6 + 2\,\theta_{34}) \wedge \eta_{14} + 3\,\eta_{12} \wedge \theta_{44} + \eta_{31} \wedge \xi^1$
$\quad + \eta_{29} \wedge \xi^2 + \eta_{30} \wedge \xi^3 + \eta_{32} \wedge \xi^4,$

$d\eta_{14} = (\eta_4 - 2\,\eta_1) \wedge \eta_{14} - 3\,\eta_3 \wedge \eta_{12} + \eta_{32} \wedge \xi^1 + \eta_{30} \wedge \xi^2,$

$d\eta_{15} = \frac{1}{3}(2\,\eta_5 - 5\,\eta_4 - 2\,\eta_1) \wedge \eta_{15} - 2\,\eta_2 \wedge \eta_{17} - \eta_3 \wedge \eta_{16} + 2\,\eta_6 \wedge \eta_7 + \eta_{26} \wedge \theta_3 - \eta_{25} \wedge \theta_4$
$\quad + \eta_{10} \wedge \theta_{13} + \eta_9 \wedge \theta_{14} - 2\,\eta_{12} \wedge \theta_{23} - 2\,\eta_{11} \wedge \theta_{33} + 3\,\eta_7 \wedge \theta_{34} + \eta_8 \wedge \theta_{44} + \eta_{33} \wedge \xi^1$
$\quad + \eta_{34} \wedge \xi^2 + \eta_{24} \wedge \xi^3 + \eta_{23} \wedge \xi^4,$

$d\eta_{16} = \frac{1}{3}\,(\eta_1 - 8\,\eta_4 + 2\,\eta_5) \wedge \eta_{16} - 3\,\eta_2 \wedge \eta_{15} + 2\,\eta_6 \wedge \eta_8 + \eta_{25} \wedge \theta_3 - \eta_{28} \wedge \theta_4 + 3\,\eta_9 \wedge \theta_{13}$
$\quad - 3\,\eta_{10} \wedge \theta_{23} + 3\,\eta_7 \wedge \theta_{33} + 5\,\eta_8 \wedge \theta_{34} + \eta_{34} \wedge \xi^1 + \eta_{35} \wedge \xi^2 + \eta_{27} \wedge \xi^3 + \eta_{24} \wedge \xi^4,$

$d\eta_{17} = \frac{1}{3}\,(2\,\eta_5 - 5\,\eta_1 - 2\,\eta_4) \wedge \eta_{17} - \eta_2 \wedge \eta_{18} - 2\,\eta_3 \wedge \eta_{15} + 2\,\eta_6 \wedge \eta_{11} + \eta_{30} \wedge \theta_3$
$\quad - \eta_{26} \wedge \theta_4 - \eta_{12} \wedge \theta_{13} + 2\,\eta_{10} \wedge \theta_{14} - \eta_{14} \wedge \theta_{23} - \eta_{13} \wedge \theta_{33} + \eta_{11} \wedge \theta_{34} + 2\,\eta_7 \wedge \theta_{44}$
$\quad + \eta_{36} \wedge \xi^1 + \eta_{33} \wedge \xi^2 + \eta_{23} \wedge \xi^3 + \eta_{29} \wedge \xi^4,$

$d\eta_{18} = \frac{1}{3}\,(\eta_4 - 8\,\eta_1 + 2\,\eta_5) \wedge \eta_{18} - 3\,\eta_3 \wedge \eta_{17} + 2\,\eta_6 \wedge \eta_{13} + \eta_{32} \wedge \theta_3 - \eta_{30} \wedge \theta_4$
$\quad - 3\,\eta_{14} \wedge \theta_{13} + 3\,\eta_{12} \wedge \theta_{14} - \eta_{13} \wedge \theta_{34} + 3\,\eta_{11} \wedge \theta_{44} + \eta_{37} \wedge \xi^1 + \eta_{36} \wedge \xi^2 + \eta_{29} \wedge \xi^3$
$\quad + \eta_{31} \wedge \xi^4,$

$d\eta_{19} = (\eta_5 - \eta_1 - 2\,\eta_4) \wedge \eta_{19} - 2\,\eta_2 \wedge \eta_{21} - \eta_3 \wedge \eta_{20} + 3\,\eta_6 \wedge \eta_{15} - \eta_{25} \wedge \theta_1 + \eta_{26} \wedge \theta_2$
$\quad + \eta_{23} \wedge \theta_3 - \eta_{24} \wedge \theta_4 + \eta_9 \wedge \theta_{11} + \eta_{10} \wedge \theta_{12} + \eta_7 \wedge \theta_{13} + \eta_8 \wedge \theta_{14} - 2\,\eta_{12} \wedge \theta_{22}$

$$
\begin{aligned}
&\quad -2\,\eta_{11}\wedge\theta_{23}-2\,\eta_{17}\wedge\theta_{33}+4\,\eta_{15}\wedge\theta_{34}+\eta_{16}\wedge\theta_{44}+\eta_{38}\wedge\xi^1+\eta_{39}\wedge\xi^2+\eta_{34}\wedge\xi^3\\
&\quad +\eta_{33}\wedge\xi^4,\\
d\eta_{20} &= (\eta_5-3\,\eta_4)\wedge\eta_{20}-3\,(\eta_2\wedge\eta_{19}-\eta_6\wedge\eta_{16})-\eta_{28}\wedge\theta_1+\eta_{25}\wedge\theta_2+\eta_{24}\wedge\theta_3\\
&\quad -\eta_{27}\wedge\theta_4+3\,(\eta_9\wedge\theta_{12}+\eta_8\wedge\theta_{13}-\eta_{10}\wedge\theta_{22}-\eta_7\wedge\theta_{23}-\eta_{15}\wedge\theta_{33}+2\,\eta_{16}\wedge\theta_{34})\\
&\quad +\eta_{39}\wedge\xi^1+\eta_{40}\wedge\xi^2+\eta_{35}\wedge\xi^3+\eta_{34}\wedge\xi^4,\\
d\eta_{21} &= (\eta_5-2\,\eta_1-\eta_4)\wedge\eta_{21}-\eta_2\wedge\eta_{22}-2\,\eta_3\wedge\eta_{19}+(3\,\eta_6-2\,\theta_{34})\wedge\eta_{17}-\eta_{26}\wedge\theta_1\\
&\quad +\eta_{30}\wedge\theta_2+\eta_{29}\wedge\theta_3-\eta_{23}\wedge\theta_4+2\,\eta_{10}\wedge\theta_{11}-\eta_{12}\wedge\theta_{12}-\eta_{11}\wedge\theta_{13}+2\,\eta_7\wedge\theta_{14}\\
&\quad -\eta_{14}\wedge\theta_{22}-\eta_{13}\wedge\theta_{23}-\eta_{18}\wedge\theta_{33}+2\,\eta_{15}\wedge\theta_{44}+\eta_{41}\wedge\xi^1+\eta_{38}\wedge\xi^2+\eta_{33}\wedge\xi^3\\
&\quad +\eta_{36}\wedge\xi^4,\\
d\eta_{22} &= (\eta_5-3\,\eta_1)\wedge\eta_{22}-3\,\eta_3\wedge\eta_{21}+3\,\eta_6\wedge\eta_{18}-\eta_{30}\wedge\theta_1+\eta_{32}\wedge\theta_2+\eta_{31}\wedge\theta_3\\
&\quad -\eta_{29}\wedge\theta_4+3\,(\eta_{12}\wedge\theta_{11}-\eta_{14}\wedge\theta_{12}-\eta_{13}\wedge\theta_{13}+\eta_{11}\wedge\theta_{14}+\eta_{17}\wedge\theta_{44})\\
&\quad +\eta_{42}\wedge\xi^1+\eta_{41}\wedge\xi^2+\eta_{36}\wedge\xi^3+\eta_{37}\wedge\xi^4.
\end{aligned}
\tag{5}
$$

For these equations, the non-zero reduced Cartan's characters are $s_1^\prime=16$ and $s_2^\prime=4$, the degree of indeterminancy is $r^{(2)}=24$, therefore Eqs. (5) are involutive, and diffeomorphisms from the symmetry pseudo-group depend on 4 arbitrary functions of two variables.

In the next calculations we use the following Maurer–Cartan forms only:

$$
\begin{aligned}
\theta_0 &= b_3^3\,b_0\,\vartheta_0,\\
\theta_1 &= b_3^3\,(b_{22}\,\vartheta_1-b_{21}\,\vartheta_2+(b_{22}\,u_{zz}-b_{21}\,(u_{yz}+b_4))\,\vartheta_3+(b_{21}\,u_{yy}-b_{22}\,(u_{yz}-b_4))\,\vartheta_4)\,,\\
\theta_2 &= b_3^3\,(-b_{12}\,\vartheta_1+b_{11}\,\vartheta_2+(b_{11}\,(u_{yz}+b_4)-b_{12}\,u_{zz})\,\vartheta_3+(b_{12}\,(u_{yz}-b_4)-b_{11}\,u_{yy})\,\vartheta_4)\,,\\
\theta_3 &= b_3^2\,(-b_{11}\,\vartheta_3+b_{12}\,\vartheta_4)\,,\\
\theta_4 &= b_3\,(b_{21}\,\vartheta_3-b_{22}\,\vartheta_4)\,,\\
\xi^1 &= b_{11}\,dt+b_{12}\,dx,\\
\xi^2 &= b_{21}\,dt+b_{22}\,dx,\\
\xi^3 &= b_3\,((b_{22}\,u_{zz}-b_{21}\,(u_{yz}-b_4))\,dt+(b_{22}\,(u_{yz}-b_4)-b_{21}\,u_{yy})\,dx-b_{22}\,dy-b_{21}\,dz)\,,\\
\xi^4 &= b_3\,((b_{11}\,(u_{yz}-b_4)-b_{12}\,u_{zz})\,dt+(b_{12}\,(u_{yz}+b_4)-b_{11}\,u_{yy})\,dx-b_{12}\,dy-b_{11}\,dz)\,,\\
\theta_{33} &= \frac{b_3}{b_0}\,\left(b_{11}^2\,\bar{\vartheta}_{33}-2\,b_{11}b_{12}\,\bar{\vartheta}_{34}+b_{12}^2\,\bar{\vartheta}_{44}\right),\\
\theta_{34} &= -\frac{b_3}{b_0}\,\left(b_{11}\,b_{21}\,\bar{\vartheta}_{33}-(b_{11}b_{22}+b_{12}b_{21})\,\bar{\vartheta}_{34}+b_{12}b_{22}\,\bar{\vartheta}_{44}\right),
\end{aligned}
$$

$$\theta_{44} = \frac{b_3}{b_0}\,\left(b_{21}^2\,\bar{\vartheta}_{33} - 2\,b_{21}b_{22}\,\bar{\vartheta}_{34} + b_{22}^2\,\bar{\vartheta}_{44}\right),$$

$$\eta_1 = \frac{1}{b_0}\,(b_{22}\,db_{11} - b_{21}\,db_{12}) - \frac{1}{b_0^2}\,\big((b_{11}b_{21}^2\,u_{yyy} - b_{21}\,(2\,b_{11}b_{22} + b_{12}b_{21})\,u_{yyz} - b_{12}b_{22}^2\,u_{zzz}$$

$$+b_{22}\,(b_{11}b_{22} + 2\,b_{12}b_{21})\,u_{yzz})\,\xi^1 - (b_{11}^2b_{21}\,u_{yyy} - b_{11}\,(b_{11}b_{22} + 2\,b_{12}b_{21})\,u_{yyz}$$

$$+b_{12}\,(2\,b_{11}b_{22} + b_{12}b_{21})\,u_{yzz} - b_{12}^2b_{22}\,u_{zzz})\,\xi^2\big),$$

$$\eta_2 = \frac{1}{b_0}\,(b_{11}\,db_{12} - b_{12}\,db_{11}) + \frac{1}{b_0^2}\,\big((b_{11}^2b_{21}\,u_{yyy} - b_{11}\,(b_{11}b_{22} + 2\,b_{12}b_{21})\,u_{yyz} - b_{12}^2b_{22}\,u_{zzz}$$

$$+b_{12}\,(2\,b_{11}b_{22} + b_{12}b_{21})\,u_{yzz})\,\xi^1 - (b_{11}^3\,u_{yyy} - 3\,b_{11}^2b_{12}\,u_{yyz} + 3\,b_{11}b_{12}^2\,u_{yzz}$$

$$-b_{12}^3\,u_{zzz})\,\xi^2\big),$$

$$\eta_3 = \frac{1}{b_0}\,\left(b_{22}db_{21} - b_{22}^2db_{11} - b_{21}db_{22} + b_{21}b_{22}db_{12}\right) + \frac{b_{22}}{b_{12}}\,\eta_1$$

$$+\frac{1}{b_{12}b_0}\,\big((b_{21}^2\,u_{yyy} - 2\,b_{21}b_{22}\,u_{yyz} + b_{22}^2\,u_{yzz})\,\xi^1$$

$$-(b_{11}b_{21}\,u_{yyy} - (b_{11}b_{22} + b_{12}b_{21})\,u_{yyz} + b_{12}b_{22}\,u_{yzz})\,\xi^2\big),$$

$$\eta_4 = \frac{1}{b_0}\,(b_{12}b_{21}\,db_{11} - b_{21}\,db_{12} - b_{12}\,db_{21} + b_{11}\,db_{22}) + \frac{b_{21}}{b_{11}}\,\eta_2$$

$$+\frac{1}{b_{11}b_0}\,\big(((b_{11}b_{22} + b_{12}b_{21})\,u_{yzz} - b_{11}b_{21}\,u_{yyz} - b_{12}b_{22}\,u_{zzz})\,\xi^1$$

$$+(b_{11}^2\,u_{yyz} - 2\,b_{11}b_{12}\,u_{yzz} + b_{12}^2u_{zzz})\,\xi^2\big),$$

$$\eta_5 = 3\left(\frac{db_3}{b_3} + \frac{1}{b_{11}b_0}\,(b_{12}b_{21}db_{11} - b_{11}b_{21}db_{12} - b_{11}b_{12}db_{21} + b_{11}^2db_{22}) + \frac{b_{21}}{b_{11}}\,\eta_2\right) + \eta_1$$

$$-2\,\eta_4 + \frac{3}{b_{11}b_0}\,\big(((b_{11}b_{22} + b_{12}b_{21})\,u_{yzz} - b_{11}b_{21}\,u_{yyz} - b_{12}b_{22}\,u_{zzz})\,\xi^1$$

$$+(b_{11}^2\,u_{yyz} - 2\,b_{11}b_{12}\,u_{yzz} + b_{12}^2\,u_{zzz})\,\xi^2\big),$$

$$\eta_6 = -b_3\,db_4 + \theta_{34}. \tag{6}$$

In these forms, $\bar{\vartheta}_{ij} = \iota_0^*\,\vartheta_{ij}$ and b_{11}, b_{12}, b_{21}, b_{22}, b_3, b_4 are arbitrary parameters such that $b_0 = b_{11}b_{22} - b_{12}b_{21} \neq 0$ and $b_{11}\,b_3 \neq 0$.

3. Contact integrable extensions

To apply Éli Cartan's structure theory of Lie pseudo-groups to the problem of finding zero-curvature representations we use the notion of integrable extension. It was introduced in [2] for the case of PDEs with two independent variables and finite-dimensional coverings. The generalization of the definition to the case of infinite-dimensional coverings of PDEs with more than two independent variables is proposed in [24]. In contrast to [30, 2], the starting point of our definition is the set of Maurer–Cartan forms of the symmetry pseudo-group of a given PDE, and all the constructions are carried out in terms of invariants of the pseudo-group. Therefore, the effectiveness of our method increases when it is applied to equations with large symmetry pseudo-groups.

Let $\mathfrak{G}$ be a Lie pseudo-group on a manifold M. Let ω^1, ... , ω^m, $m = \dim M$, be its Maurer–Cartan forms with the structure equations

$$d\omega^i = A^i_{\gamma j}\, \pi^\gamma \wedge \omega^j + B^i_{jk}\, \omega^j \wedge \omega^k, \tag{7}$$

where $\gamma \in \{1, \dots, \Gamma\}$ for some $\Gamma \geq 0$. The coefficients $A^i_{\gamma j}$, $B^i_{jk} = -B^i_{kj}$ in these equations depend on the invariants U^κ, $\kappa \in \{1, \dots, \Lambda\}$, $\Lambda \geq 0$. The differentials of the invariants satisfy equations

$$dU^\lambda = C^\lambda_j\, \omega^j, \tag{8}$$

where C^λ_j are functions of U^κ. Consider the following system of equations

$$\begin{aligned} d\tau^q &= D^q_{\rho r}\, \eta^\rho \wedge \tau^r + E^q_{rs}\, \tau^r \wedge \tau^s + F^q_{r\beta}\, \tau^r \wedge \pi^\beta + G^q_{rj}\, \tau^r \wedge \omega^j + H^q_{\beta j}\, \pi^\beta \wedge \omega^j \\ &\quad + I^q_{jk}\, \omega^j \wedge \omega^k, \end{aligned} \tag{9}$$

$$dV^\epsilon = J^\epsilon_j\, \omega^j + K^\epsilon_q\, \tau^q, \tag{10}$$

for unknown 1-forms τ^q, $q \in \{1, ..., Q\}$, η^ρ, $\rho \in \{1, ..., R\}$, and unknown functions V^ϵ, $\epsilon \in \{1, ..., S\}$ with some $Q, R, S \in \mathbb{N}$. The coefficients $D^\kappa_{\rho r}$, ..., K^ϵ_q in Eqs. (9), (10) are supposed to be fucntions of U^λ and V^γ.

DEFINITION 1. The system (9), (10) is called an *integrable extension* of the system (7), (8), if Eqs. (9), (10), (7) (8) together meet the involutivity conditions and the compatibility conditions

$$d(d\tau^q) \equiv 0, \qquad d(dV^\epsilon) \equiv 0. \tag{11}$$

Eqs. (11) give an over-determined system of PDEs for the coefficients $D^\kappa_{\rho r}$, ..., K^ϵ_q in Eqs. (9), (10). If this system is satisfied, the third inverse fundamental Lie's theorem in Cartan's form, [3, §§16, 22–24], [6], [29, §§16, 19, 20, 25,26], [28, §§14.1–14.3], ensures the existence of the forms τ^q, V^ϵ, the solutions to Eqs. (9), (10). In acccordance with the second inverse fundamenatl Lie's theorem, the forms τ^q, ω^i are Maurer–Cartan forms for a Lie pseudo-group $\mathfrak{H}$ acting on $M \times \mathbb{R}^Q$.

DEFINITION 2. The integrable extension (9), (10) is called *trivial*, if there exists a change of variables on the manifold of action of the pseudo-group $\mathfrak{H}$ such that in the new coordinates the coefficients $F^q_{r\beta}$, G^q_{rj}, $H^q_{\beta j}$, I^q_{jk} and J^ϵ_j are identically equal to zero,

while the coefficients $D^q_{\rho r}$, E^q_{rs} and K^ϵ_q are independent of U^λ. Otherwise, the integrable extension is called *nontrivial*.

Let θ^α_I and ξ^j be a set of Maurer–Cartan forms of a symmetry pseudo-group $\mathfrak{Lie}(\mathcal{E})$ of a PDE $\mathcal{E}$ such that ξ^i are horizontal forms, that is, $\xi^1 \wedge ... \wedge \xi^n \neq 0$ on each solution of $\mathcal{E}$, while θ^α_I are contact forms, that is, they are equal to 0 on each solution.

DEFINITION 3. Nontrivial integrable extension of the structure equations for the pseudo-group $\mathfrak{Lie}(\mathcal{E})$ of the form

$$d\omega^q = \Pi^q_r \wedge \omega^r + \xi^j \wedge \Omega^q_j, \tag{12}$$

$q, r \in \{1, \dots, N\}$, $N \geq 1$, is called a *contact integrable extension*, if the follownig conditions are satisfied:

(*i*) $\Omega^q_j \in \langle \theta^\alpha_I,\, \omega^r_i \rangle_{\texttt{lin}}$ for some additional 1-forms ω^r_i;

(*ii*) $\Omega^q_j \notin \langle \omega^r_i \rangle_{\texttt{lin}}$ for some q and j;

(*iii*) $\Omega^q_j \notin \langle \theta^\alpha_I \rangle_{\texttt{lin}}$ for some q and j;

(*iv*) $\Pi^q_r \in \langle \theta^\alpha_I,\, \xi^j,\, \omega^r,\, \omega^r_i \rangle_{\texttt{lin}}$.

(*v*) The coefficients of expansions of the forms Ω^q_j with respect to $\{\theta^\alpha_I,\, \omega^r_i\}$ and the forms Π^q_r with respect ot $\{\theta^\alpha_I,\, \xi^j,\, \omega^r,\, \omega^r_i\}$ depend either on the invariants of the pseudo-group $\mathfrak{Lie}(\mathcal{E})$ alone, or they depend also on a set of some additional functions W_ρ, $\rho \in \{1, \dots, \Lambda\}$, $\Lambda \geq 1$. In the latter case, there exist functions $P^{I\rho}_\alpha$, Q^ρ_q, $R^{j\rho}_q$ and S^ρ_j such that

$$dW_\rho = P^I_{\rho\alpha}\, \theta^\alpha_I + Q_{\rho q}\, \omega^q + R^j_{\rho q}\, \omega^q_j + S_{\rho j}\, \xi^j, \tag{13}$$

and the set of equations (13) satisfies the compatibility conditions

$$d(dW_\rho) = d\left(P^I_{\rho\alpha}\, \theta^\alpha_I + Q_{\rho q}\, \omega^q + R^j_{\rho q}\, \omega^q_j + S_{\rho j}\, \xi^j\right) \equiv 0. \tag{14}$$

We apply this definition to the structure equations (5). We restrict our analysis to CIEs of the form

$$\begin{aligned} d\omega_0 = {} & \left(\sum_{i=0}^{4} A_i\, \theta_i + \sum{}^* B_{ij}\, \theta_{ij} + \sum_{s=1}^{22} C_s\, \eta_s + \sum_{j=1}^{4} D_j\, \xi^j + \sum_{k=1}^{2} E^k\, \omega_k\right) \wedge \omega_0 \\ & + \sum_{k=1}^{4}\left(\sum_{i=0}^{4} F_{ik}\, \theta_i + \sum{}^* G_{ijk}\, \theta_{ij} + \sum_{m=1}^{2} H^m_k\, \omega_m\right) \wedge \xi^k, \end{aligned} \tag{15}$$

with two additional forms ω_1 and ω_2 mentioned in the part (i) of Definition 3. In (15), $\sum{}^*$ means summation for all $i, j \in \mathbb{N}$ such that $1 \leq i \leq j \leq 4$, $(i, j) \neq (2, 4)$. These equations together with Eqs. (5) satisfy the requirement of involutivity. We assume that the coefficients of (15) are either constants or functions of additional invariants W_ρ mentioned in the part (v) of Definition 3. In the latter case the differentials of W_ρ meet the following requirement

$$dW_\rho = \sum_{i=0}^{4} I_{\rho i}\, \theta_i + \sum{}^* J_{\rho ij}\, \theta_{ij} + \sum_{s=22}^{7} K_{\rho s}\, \eta_s + \sum_{j=1}^{4} L_{\rho j}\, \xi^j + \sum_{q=0}^{2} M_{\rho q}\, \omega_q. \tag{16}$$

Defintion 3 yields an over-determined system for the coefficients of (15) and (16). The results of analysis of this system are summarized in the following theorem.

THEOREM 1. *There are no* CIE*s (15) with constant coefficients or* CIE*s (15), (16) with one additional invariant W_1. Every* CIE *(15), (16) with two additional invariants W_1, W_2 is contact-equivalent either to*

$$d\omega_0 = \big(\omega_1 + W_1\,\eta_2 + \tfrac{1}{3}\,(\eta_5 + 2\,\eta_4 - \eta_1)\big) \wedge \omega_0 + (W_1\,\theta_{34} - \theta_{44} + W_2\,\omega_2) \wedge \xi^1$$
$$+ (W_1\,\theta_{33} - \theta_{34} + W_2\,\omega_1) \wedge \xi^2 + \omega_1 \wedge \xi^3 + \omega_2 \wedge \xi^4, \tag{17}$$
$$dW_1 = W_1\,\omega_1 - \omega_2 - W_1\,\eta_1 + W_1^2\,\eta_2 - \eta_3 + W_1\,\eta_4 + Z_1\,(\omega_0 + W_2\,\xi^2 + \xi^3)$$
$$+ Z_2\,(W_2\,\xi^1 + \xi^4), \tag{18}$$
$$dW_2 = \eta_6 - \theta_{34} + \tfrac{1}{3}\,W_2\,(\eta_5 - \eta_1 - \eta_4) + Z_3\,(\omega_0 + W_2\,\xi^2 + \xi^3) + Z_4\,(W_2\,\xi^1 + W_1\,\xi^4) \tag{19}$$

or to

$$d\omega_0 = \big(\omega_2 + W_1\,\eta_3 + \tfrac{1}{3}\,(\eta_5 + 2\,\eta_1 - \eta_4)\big) \wedge \omega_0 + (\theta_{34} - W_1\,\theta_{44} + W_2\,\omega_2) \wedge \xi^1$$
$$+ (\theta_{33} - W_1\,\theta_{34} + W_2\,\omega_1) \wedge \xi^2 + \omega_1 \wedge \xi^3 + \omega_2 \wedge \xi^4, \tag{20}$$
$$dW_1 = W_1\,\omega_2 - \omega_1 + W_1\,\eta_1 - \eta_2 + W_1^2\,\eta_3 - W_1\,\eta_4 + Z_1\,(\omega_0 + W_2\,\xi^1 + \xi^4)$$
$$+ Z_2\,(W_2\,\xi^1 + \xi^4), \tag{21}$$
$$dW_2 = \eta_6 - \theta_{34} + \tfrac{1}{3}\,W_2\,(\eta_5 - \eta_1 - \eta_4) + Z_3\,(\omega_0 + W_2\,\xi^1 + \xi^4) + Z_4\,(W_2\,\xi^2 + W_1\,\xi^3), \tag{22}$$

where Z_1, ... , Z_4 are arbitrary parameters.

The forms (6) in Eqs. (17), (18), (19) and Eqs. (20), (21), (22) are known explicitly, therefore, in accordance with the third inverse fundamental Lie's theorem, the forms ω_0 satisfying (17) or (20) can by found by means of integration. This analysis splits into two cases — when $Z_3 = 0$ or $Z_3 \neq 0$.

REMARK 1. When $Z_3 = 0$ in Eq. (19) or Eq. (22), the functions W_2 appear to be independent of the fibre coordinates of the covering. This entails that one symmetry of Eq. (1) is unliftable to the fibre of the covering. From results of [17, 14, 11, 12, 19] it follows that the corresponding covering has a non-removable parameter. Thus the appearance of the non-removable parameter in the covering can be deduced from the form of the CIE directly, before integration of its equations.

The results of integration of Eqs. (17), (18), (19) and Eqs. (20), (21), (22) are given in the following theorem.

THEOREM 2. *When $Z_3 = 0$, every solution to Eq. (17) up to a contact equivalence is*

$$\omega_0 = \frac{b_0 b_3}{b_{12}\,v_z - b_{11}\,v_y}\,\left(dv + (v_{zz}\,v_y - (u_{yz} + \lambda)\,v_z)\,dt + ((u_{yz} - \lambda)\,v_y - u_{yy}\,v_z)\,dx\right.$$
$$\left.-v_y\,dy - v_z\,dz\right), \tag{23}$$

whereas for $Z_3 \neq 0$ *it is*

$$\omega_0 = \frac{b_0 b_3}{b_{12}\, v_z - b_{11}\, v_y}\, \left(dv + (v_{zz}\, v_y - (u_{yz} + v)\, v_z)\, dt + ((u_{yz} - v)\, v_y - u_{yy}\, v_z)\, dx \right.$$
$$\left. -v_y\, dy - v_z\, dz\right). \qquad (24)$$

The solutions to Eq. (20) can be obtained from (23) and (24) by the following simple change of independent variables: $(t, x, y, z) \mapsto (x, t, z, y)$.

When we put $\omega_0 = 0$, Eq. (23) gives the system (2), while Eq. (24) defines new covering

$$\begin{cases} v_t = (u_{yz} + v)\, v_z - u_{zz}\, v_y, \\ v_x = u_{yy}\, v_z - (u_{yz} - v)\, v_y \end{cases}$$

for the second heavenly equation. These equations are nonlinear w.r.t. the fibre variable v.

REMARK 2. Direct computation shows that the symmetry of Eq. (1) with the infinitesimal generator $X = t\, \frac{\partial}{\partial y} + x\, \frac{\partial}{\partial z}$ is unliftable to a symmetry of Eqs. (2). Since $e^{\lambda X}(u_{yy}, u_{yz}, u_{zz}) = (u_{yy}, u_{yz}, u_{zz})$ and $e^{\lambda X}(v, v_t, v_x, v_y, v_z) = (v, v_t{+}\lambda\, v_y, v_x{+}\lambda\, v_z, v_y, v_z)$, the parameter λ in Eqs. (2) can be obtained by the action of $e^{\lambda X}$ to the system (2) with $\lambda = 0$. Therefore, λ is the non-removable parameter of the covering (2).

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